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Bernstein Type Theorems for Compact Sets in $\mathbb{R}^n$

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In this paper we give two non-trivial generalizations of a classical Bernstein inequality which is apparently less known that that of Bernstein–Markov, viz.

$$|p'(x)| \leqslant k(1-x^2)^{-1/2}\,(\|p\|^2_{[-1,1]} - p^2(x))^{1/2},$$

for $x \in (-1, 1)$, where p is a real polynomial of $\deg p \leqslant k$ and $\|p\|_{[-1,1]} = \sup|p|([-1,1])$, to the case of a compact set E in $\mathbb{R}^n$ with nonempty interior. Contrary to the situation where estimates for $p'(x)$ are sought on the whole compact set, we do not, in general, need any other assumptions on E. Our results point out connections between Bernstein's inequality and two important notions in modern polynomial approximation theory on compacta in $\mathbb{C}^n$: Siciak's extremal function and complex equilibrium measure.

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Introduction and Statement of the Main Results. We start with some classical inequalities for polynomials: the Bernstein–Markov inequality (see [11])

$$(1.1) \qquad |p'(x)| \leqslant k(1-x^2)^{-1/2}\,\|p\|_{[-1,1]}, \qquad \text{for} \quad x \in (-1, 1),$$

and the Bernstein inequality (see [9])

$$(1.2) \qquad |p'(x)| \leqslant k(1-x^2)^{-1/2}\,(\|p\|^2_{[-1,1]} - p^2(x))^{1/2}, \qquad \text{for} \quad x \in (-1, 1),$$

where p is a real polynomial with $\deg p \leqslant k$. It is easily seen that (1.1) implies

$$(1.3) \qquad \int_{-1}^{1} |p'(x)|\, dx \leqslant \pi k \|p\|_{[-1,1]}.$$

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The main goal of this paper is to prove analogous results in the multivariate case. Let us begin with some definitions and facts from complex analysis of several variables.

If E is a compact subset of $\mathbb{C}^n$ $(n \geqslant 1)$ then we define Siciak's extremal function Φ_E as follows (see [19]).

1.4. Definition. $\Phi_E(z) = \sup(|p(z)|^{1/\deg p}\colon p \in \mathbb{C}[w_1, ..., w_n],\ \deg p \geqslant 1,$ $\|p\|_E \leqslant 1\}$, for $z \in \mathbb{C}^n$, where $\|p\|_E = \sup|p|(E)$. The above extremal function is also called the polynomial extremal function as opposed to the plurisubharmonic extremal function V_E and its upper regularization V_E^* defined as follows.

1.5. Definition. $V_E(z) = \sup\{u(z)\colon u \in \mathscr{L}_n,\ u|_E \leqslant 0\}$, for $z \in \mathbb{C}^n$, where $\mathscr{L}_n$ denotes the Lelong class of plurisubharmonic functions in $\mathbb{C}^n$ (briefly, $\mathrm{PSH}(\mathbb{C}^n)$) with logarithmic growth: $\mathscr{L}_n = \{u \in \mathrm{PSH}(\mathbb{C}^n)\colon$ $\sup\{u(z) - \log(1 + |z|)\colon z \in \mathbb{C}^n\} < \infty\}$.

$$(1.6) \qquad V_E^*(z) = \operatorname*{lum\,sup}_{w \to z} V_E(w), \qquad z \in \mathbb{C}^n.$$

The crucial fact is that

1.7. Zachariuta–Siciak Theorem (see [23, 20]). $V_E = \log \Phi_E$.

For other properties of the extremal functions we refer the reader to Siciak's papers [20, 21]. We will need the notions of L-regularity and pluripolarity.

1.8. Definition. We call a compact set E L-regular at a point $a \in E$ if $V_E^*(a) = 0$ and we say that E is L-regular if E is L-regular at every point $a \in E$. It is known (see [20, 23]) that E is L-regular if and only if V_E is continuous in $\mathbb{C}^n$. Often, it is possible to use the following geometrical criterion for the L-regularity.

1.9. Proposition (Cegrell [10], Plesniak [17], Sadullaev [18]). *Given $a \in E$, suppose that there exists an analytic mapping $h\colon [0, 1] \to E$ such that $h(0) = a$. If $V_E^*(h(t)) = 0$ for each $t \in (0, 1]$ then $V_E^*(a) = 0$.*

A pluripolar set is defined as follows.

1.10. Definition. We call a set E pluripolar if there exists a function $u \in \mathrm{PSH}(\mathbb{C}^n)$ such that $E \subset \{u = -\infty\}$.

If a compact set E is not pluripolar then $V_E^* \in \mathrm{PSH}(\mathbb{C}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{C}^n)$ (see

[20]). In this case we define the complex equilibrium measure λ_E as the value of the complex Monge–Ampère operator on the function V_E^*.

1.11. DEFINITION. $\lambda_E=(dd^c V_E^*)^n$. Then λ_E is a Borel measure on $\mathbb{C}^n$ (for details we refer to Bedford and Taylor's paper [6]). We note that if $u\in \mathrm{PSH}\cap C^2(\Omega)$ then $(dd^c u)^n$ is a Borel measure defined by

$$(dd^c u)^n = n!\,4^n \det\left[\frac{\partial^2 u}{\partial z_i\,\partial \bar{z}_j}\right] dV_n(z),$$

where V_n is the Lebesgue measure in $\mathbb{C}^n$. The main properties of the complex equilibrium measure are contained in the following.

1.12. PROPOSITION [6, 22]. *If E is a non-pluripolar compact set in $\mathbb{C}^n$, then*

$$\lambda_E(\mathbb{C}^n\setminus\hat{E})=0, \qquad \lambda_E(\hat{E})=(2\pi)^n,$$

where $\hat{E}=\{z\in\mathbb{C}^n\colon |p(z)|\leqslant \|p\|_E$ for each $p\in\mathbb{C}[w_1,...,w_n]\}$.

We now may formulate our main results. Let E be a compact set in $\mathbb{R}^n$. We regard here $\mathbb{R}^n$ as a subset of $\mathbb{C}^n$ such that $\mathbb{C}^n=\mathbb{R}^n+i\mathbb{R}^n$. We need the following definition.

1.13. DEFINITION. If E is a compact subset of $\mathbb{R}^n$ then we put

$$D_j^+ V_E(x)=\varliminf_{\varepsilon\to 0+}\frac{1}{\varepsilon} V_E(x+i\varepsilon e_j)$$

and

$$\operatorname{grad}_+ V_E(x)=\left(\varliminf_{\varepsilon\to 0+}\frac{1}{\varepsilon}V_E(x+i\varepsilon e_1),...,\varliminf_{\varepsilon\to 0+}\frac{1}{\varepsilon}V_E(x+i\varepsilon e_n)\right),$$

for $x\in E$, where $\{e_1,...,e_n\}$ is the standard orthonormal basis in $\mathbb{R}^n$.

1.1.4. THEOREM. *Let E be a compact set in $\mathbb{R}^n$ with nonempty interior. Then for every $x\in\operatorname{int}(E)$ we have the following inequality for a real polynomial p*

$$|D_j p(x)|\leqslant(\deg p)\,D_j^+V_E(x)(\|p\|_E^2-p^2(x))^{1/2}\qquad \textit{for}\quad j=1,...,n$$

and

$$|\operatorname{grad} p(x)|\leqslant(\deg p)\,|\operatorname{grad}_+ V_E(x)|\,(\|p\|_E^2-p^2(x))^{1/2}.$$

1.15. THEOREM. *Let E be a compact L-regular set in* $\mathbb{R}^n$ *with nonempty interior. Then the measure* $\lambda_E|_{\mathrm{int}(E)}$ *is absolutely continuous with respect to the Lebesgue measure and*

$$\mathrm{vol}\left(\mathrm{conv}\left\{\frac{1}{\deg p}(1-p^2(x))^{-1/2}\,\mathrm{grad}\,p(x) \colon p\in\mathbb{R}[z],\ \deg p\geqslant 1,\right.\right.$$

$$\left.\left.\|p\|_E\leqslant 1 \text{ and } |p(x)|<1 \text{ on } \mathrm{int}(E)\right)\right\}\leqslant\frac{1}{n!}\,\lambda(x).$$

for almost every $x\in\mathrm{int}(E)$ *(with respect to the Lebesgue measure), where* $\lambda(x)\,dx=\lambda_E|_{\mathrm{int}(E)}$. *If* $n=1$, *then the above equality reduces to*

$$\sup\left\{\frac{1}{\deg p}(1-p^2(x))^{-1/2}\,|p'(x)| \colon p\in\mathbb{R}[z],\ \deg p\geqslant 1,\right.$$

$$\left.\|p\|_E\leqslant 1 \text{ and } |p(x)|<1 \text{ on } \mathrm{int}(E)\right\}\leqslant\frac{1}{2}\,\lambda(x).$$

In this paper we prove only Theorem 1.14. It will be done in Section 2 while in Section 3 we discuss some special cases and examples to this theorem. The proof of Theorem 1.15, which we omit here (because it is more longer and difficult) will be published in a forthcoming paper [5] (see also [3]). However, in Section 4 we present some examples and applications of this theorem.

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Proof of Theorem 1.14. The proof is based on the properties of the Joukowski function and its inverse. For $z\in\mathbb{C}\setminus\{0\}$ we define the holomorphic function $g(z)=(1/2)(z+1/z)$, called the Joukowski function. It is univalent on $|z|>1$ and its inverse is of the form $h(z)=z+(z^2-1)^{1/2}$, if we choose an appropriate branch of the square root. The function $\log|h|$ is subharmonic in $\mathbb{C}$ and it is well known that

$$\Phi_{[-1,1]}(z)=|h(z)|, \qquad \text{for} \quad z\in\mathbb{C}.$$

In our considerations the crucial role is played by the following equality for the above defined function g:

$$|g(z)+1|+|g(z)-1|=2g(|z|), \qquad z\neq 0. \tag{2.1}$$

Note that every holomorphic in $\mathbb{C}\setminus\{0\}$, non-constant solution of Eq. (2.1) has a form $g((az)^p)$ with some $a>0$ and $p\in\mathbb{N}$ (see [2]). From (2.1) it follows that

$$(2.2)\qquad |h(z)| = h(\tfrac{1}{2}|z+1+\tfrac{1}{2}|z-1|),$$

for each $z\in\mathbb{C}$, where at the right-hand side we have $h(t)=t+(t^2-1)^{1/2}$ for $t\geqslant 1$ with the arithmetic root. It is easy to verify the following estimates for the function $h(t)$:

$$(2.3)\qquad \sqrt{2}\,(t-1)^{1/2}-\tfrac{1}{6}(t-1)^{3/2}\leqslant \log h(t)\leqslant \sqrt{2}\,(t-1)^{1/2}$$

for every $t\geqslant 1$. An easy computation shows that the following proposition holds.

2.4. PROPOSITION. (i) *If* $\alpha\in(-1,1)$ *and* $\varepsilon>0$, $\beta\in\mathbb{R}$, *then*

$$\frac{1}{\varepsilon}\log|h(\alpha+i\varepsilon\beta)|\leqslant|\beta|(1-\alpha^2)^{-1/2};$$

(ii) *If* $\alpha\in(-1,1)$, $0<\varepsilon\leqslant 1/2$, $\beta\in\mathbb{R}$, *and* $|\beta|\leqslant 1-|\alpha|$, *then*

$$(1-\varepsilon)\,|\beta|\,(1-\alpha^2)^{-1/2}\leqslant\frac{1}{\varepsilon}\log|h(\alpha+i\varepsilon\beta)|.$$

Consider a real polynomial p with $\|p\|_E<1$. By well-known properties of plurisubharmonic functions (see, e.g., [12]) we have $\log|h\circ p|\in\mathrm{PSH}(\mathbb{C}^n)$ and moreover, by 2.2, we have $(1/\deg p)\log|h\circ p|\in\mathscr{L}_n$. Hence, by Definition 1.5 we obtain

$$(2.5)\qquad \frac{1}{\deg p}\log|h(p(z))|\leqslant V_E(z)$$

for every $z\in\mathbb{C}^n$. Taylor's formula for p now yields

$$(2.6)\qquad p(x+i\varepsilon e_k)=p(x)+i\varepsilon D_k p(x)+\sum_{2\leqslant m\leqslant\deg p}\frac{\partial^m}{\partial x^k}p(x)(i\varepsilon)^m,$$

for $1\leqslant k\leqslant n$. It follows from Proposition 2.4 and (2.6) that

$$(2.7)\qquad \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\log|h(p(x+i\varepsilon e_k))|=|D_k p(x)|\,(1-p^2(x))^{-1/2}$$

for $x\in E$. But (2.7) together with (2.5) implies

$$(2.8)\qquad |\mathrm{grad}\,p(x)|\leqslant(\deg p)\,|\mathrm{grad}_+\,V_E(x)|\,(1-p^2(x))^{1/2}.$$

If now p is any real polynomial then we apply (2.8) to the polynomial $p/(\|p\|_E+\delta)$, and letting $\delta\to 0+$ completes the proof of Theorem 1.14.

2.9. *Remark.* If E is a compact set in $\mathbb{R}^n$ then it follows easily that $\Phi_E(z) = \sup\{|h(p(z))|^{1/\deg p}: p \in \mathbb{R}[w_1, ..., w_n], \deg p \geqslant 1, \|p\|_E \leqslant 1\}$, where h denotes, as in the whole paper, the inverse function to the Joukowski function.

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In this section we consider some special cases of Theorem 1.14. Let E be a compact, convex, and symmetric subset of $\mathbb{R}^n$ with $\operatorname{int}(E) \neq \varnothing$. By E^* we denote the dual convex set to E:

$$E^* = \{x \in \mathbb{R}^n: x \cdot y \leqslant 1 \text{ for each } y \in E\}.$$

It is known that

$$\Phi_E(z) = \sup\{|h(z \cdot w)|: w \in \partial E^*\},$$

for $z \in \mathbb{C}^n$ (see [14, 7]) and more precisely [1],

$$\Phi_E(z) = \sup\{|h(z \cdot w)|: w \in \operatorname{extr}(E^*)\}, \tag{3.1}$$

where $\operatorname{extr}(E^*)$ denotes the set of all extremal points of E^*. In the special case of $E = \bar{B}_n = \{x \in \mathbb{R}^n: x_1^2 + \cdots + x_n^2 \leqslant 1\}$ we have (see [15, 1])

$$\Phi_E(z) = (h(|z|^2 + |z^2 - 1|))^{1/2}, \qquad z \in \mathbb{C}^n,$$

where $z^2 = z_1^2 + \cdots + z_n^2$.

An easy computation shows that

$$|\operatorname{grad}_+ V_{\bar{B}_n}(x)| = (n - 1 + (1 - x^2)^{-1})^{1/2} \leqslant \sqrt{n}(1 - x^2)^{-1/2}.$$

Thus it follows from Theorem 1.14 that for each real polynomial p

$$|\operatorname{grad} p(x)| \leqslant (\deg p)(n - 1 + (1 - x^2)^{-1})^{1/2} (\|p\|_{\bar{B}_n}^2 - p^2(x))^{1/2},$$

for $|x| < 1$, which extends the Bernstein inequality (1.2).

Let now f be any norm in $\mathbb{R}^n$ and put $E = \{x \in \mathbb{R}^n: f(x) \leqslant 1\}$. It is easy to check that $f(x) = \sup\{|x \cdot w|: w \in \operatorname{extr}(E^*)\}$. Since E is compact, convex, and symmetric (with nonempty interior) it follows from (3.1) that

$$\begin{aligned} \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} V_E(x + i\varepsilon e_k) &= \sup\{|e_k \cdot w| \, (1 - (x \cdot w)^2)^{-1/2}: w \in \operatorname{extr}(E^*)\} \\ &\leqslant f(e_k)(1 - f^2(x))^{-1/2}, \end{aligned} \tag{3.2}$$

if $f(x) < 1$. This yields the following generalization of the Bernstein inequality:

3.3. PROPOSITION. *Let $E = \{f(x) \leqslant 1\}$, where f is a norm in $\mathbb{R}^n$. Then*

$$|D_j p(x)| \leqslant (\deg p)\, f(e_j)(1 - f^2(x))^{-1/2} (\|p\|_E^2 - p^2(x))^{1/2}$$

if $f(x) < 1$, where p is any real polynomial and $j = 1, ..., n$.

It is clear that $V_E \leqslant V_F$, if $F \subset E$. A trivial verification shows that if a compact set E has nonempty interior and $x \in \operatorname{int}(E)$ then

$$|\operatorname{grad}_+ V_E(x)| \leqslant \sqrt{n}/\operatorname{dist}(x, \partial E). \tag{3.4}$$

In particular, $|\operatorname{grad}_+ V_E(x)|$ is always finite if x is an interior point of E.

3.5. EXAMPLE. Let S_n be the standard simplex in $\mathbb{R}^n$:

$$S_n = \{x \in \mathbb{R}^n \colon x_1, ..., x_n \geqslant 0 \text{ and } x_1 + \cdots + x_n \leqslant 1\}.$$

Then (see [1]) we have $\Phi_{S_n}(z) = h(|z_1| + \cdots + |z_n| + |z_1 + \cdots + z_n - 1|)$ for $z \in \mathbb{C}^n$. An easy computation shows that

$$|\operatorname{grad}_+ V_{S_n}(x)| = (n(1 - x_1 - \cdots - x_n)^{-1} + 1/x_1 + \cdots + 1/x_n)^{1/2}.$$

Now we will prove an interesting version of Bernstein's inequality for convex sets in $\mathbb{R}^n$. Let E be a compact, convex subset of $\mathbb{R}^n$ with non empty interior. For simplicity assume that $0 \in \operatorname{int}(E)$. Then the following proposition holds.

3.6. PROPOSITION (see [4]). *If E is a compact, convex subset of $\mathbb{R}^n$ with $0 \in \operatorname{int}(E)$ and E^* is the convex dual set to E, then*

$$\Phi_E(z) \leqslant \inf_{d \in \operatorname{int}(E)} \sup_{w \in K} \left| h\left(\frac{(z-d) \cdot w}{1 - |d \cdot w + \beta|}\right)\right|, \qquad \text{for} \quad z \in \mathbb{C}^n,$$

where $K = (2/(1 + |\alpha|)) \operatorname{extr}(E^*)$, $\alpha = \inf\{x \cdot y \colon x \in E, y \in E^*\}$, *and* $\beta = -(1 + \alpha)/(1 + |\alpha|)$.

Now, fix $x \in \operatorname{int}(E)$. Let $d' = (1/2)(x + d)$ for any $d \in \operatorname{int}(E)$. By Proposition 3.6 we obtain

$$V_E(x + i\varepsilon e_j) \leqslant \inf_{d \in \operatorname{int}(E)} \sup_{w \in K} \log \left| h\left(\frac{(1/2)(x-d) \cdot w + i\varepsilon e_j w}{1 - |d' \cdot w + \beta|}\right)\right|.$$

Hence we get

$$
\begin{aligned}
D_j^+ V_E(x) &\leqslant \inf_{d \in \operatorname{int}(E)} \sup_{wK} |e_j \cdot w| \, (1 - |d' \cdot w + \beta|)^{-1} \\
&\quad \times \left(1 - \left(\frac{(x-d)\cdot w}{2(1-|d' \cdot w + \beta|)}\right)^2\right)^{-1/2} \\
&= \inf_{d \in \operatorname{int}(E)} \sup_{w \in K} |e_j \cdot w| \left\{(1 - |d' \cdot w + \beta|)^2 - \left(\frac{1}{2}(x-d)\cdot w\right)^2\right\}^{1/2} \\
&\leqslant \inf_{d \in \operatorname{int}(E)} \sup_{w \in K} |e_j \cdot w| (1 - |x \cdot w + \beta|)^{-1/2} (1 - |d \cdot w + \beta|)^{-1/2} \\
&\leqslant \sup_{w \in K} (|e_j \cdot w|/|w|)(\operatorname{dist}(x, \partial E))^{-1/2} \\
&\qquad \inf_{d \in \operatorname{int}(E)} (\operatorname{dist}(d, \partial E))^{-1/2}.
\end{aligned}
$$

(Here $e_1, ..., e_n$ denotes the standard orthonormal basis in $\mathbb{R}^n$.) The above inequality yields the following

3.7. Theorem. *Let E be a convex, compact subset of $\mathbb{R}^n$ and such that $0 \in \operatorname{int}(E)$. Then for every real polynomial p we have the Bernstein inequality*

$$|D_j p(x)| \leqslant (\deg p)\, M(\operatorname{dist}(x, \partial E))^{-1/2} (\|p\|_E^2 - p^2(x))^{1/2},$$

for $x \in \operatorname{int}(E)$, $j = 1, ..., n$, where the constant M is equal to

$$M = \max_{j=1,...,n} \sup_{w \in K} (|e_j \cdot w|/|w|) \inf_{d \in \operatorname{int}(E)} (\operatorname{dist}(d, \partial E))^{-1/2}.$$

3.8. *Remark.* If E is any compact, convex subset of $\mathbb{R}^n$ with nonempty interior and $b \in \operatorname{int}(E)$, then $0 \in \operatorname{int}(E-b)$ and we may apply Theorem 3.7 to the subset $E-b$. This gives the Bernstein inequality for the set E with a different constant M than that of Theorem 3.7.

3.9. *Remark.* We shall say that a compact subset E of $\mathbb{R}^n$ (with nonempty interior) has Bernstein's property if for every real polynomial p the following inequalities hold:

$$|D_j p(x)| \leqslant (\deg p)\, M(\operatorname{dist}(x, \partial E))^{-\mu} (\|p\|_E^2 - p^2(x))^{1/2}, \qquad \text{for} \quad x \in \operatorname{int}(E),$$

$j = 1, ..., n$, where $M > 0$ and $0 < \mu < 1$. Observe that every compact subset of $\mathbb{R}^n$ with nonempty interior satisfies the above inequality with the constant $\mu = 1$. We conjecture that every fat ($E \subset \overline{\operatorname{int}(E)}$) compact subset of $\mathbb{R}^n$

that preserves Bernstein's inequality with $\mu<1$ has the following Markov property: There exists a constant M such that for every real polynomial p,

$$\|D_j p\|_E \leqslant M(\deg p)^\alpha \|p\|_E, \qquad j=1, ..., n,$$

with $\alpha=1/(1-\mu)$.

We also note that the above conjecture is true in the case of compact, convex sets (see [16]).

4

In this section we will prove the following two estimates for real polynomials resulting from Theorem 1.15.

4.1. THEOREM. *Let E be an L-regular compact subset of $\mathbb{R}^n$ with noempty interior. Then for almost every $x\in \operatorname{int}(E)$ the following inequality holds*

$$|\operatorname{grad} p(x)| \leqslant 2^{-n}(\deg p)\, d(x)\, \lambda(x)(\|p\|_E^2 - p^2(x))^{1/2},$$

for a real polynomial p, where $\lambda(x)$ is the density on $\operatorname{int}(E)$ (with respect to the Lebesgue measure) of the complex equilibrium measure and

$$d(x)=[(d_1^2-x_1^2)\cdot \cdots \cdot(d_n^2-x_n^2)((d_1^2-x_1^2)^{-1}+\cdots+(d_n^2-x_n^2)^{-1})]^{1/2},$$

with $d_j=\sup|z_j|(E)$, $j=1, ..., n$.

4.2. THEOREM. *If E is a fat ($E\subset\overline{\operatorname{int}(E)}$) compact subset of $\mathbb{R}^n$ with zero Lebesgue measure on ∂E, then*

$$\int_E |\operatorname{grad} p(x)|\, dx \leqslant \pi^n(\deg p)\, d(0)\, \|p\|_E,$$

for any real polynomial p, where $d(x)$ is defined in Theorem 4.1.

4.3. *Proof of Theorem* 4.1. Without loss of generality we can assume $|p(x)|<\|p\|_E$ for $x\in\operatorname{int}(E)$. From Theorem 1.15 it follows that

$$\begin{aligned}
&n!\operatorname{vol}(\operatorname{conv}\{\pm(\|p\|_E^2-p^2(x))^{-1/2}\operatorname{grad} p(x),\\
&\qquad \pm(d_1^2-x_1^2)^{-1/2}e_1, ..., \pm(d_i^2-x_i^2)^{-1/2}\hat{e}_i, ..., \pm(d_n^2-x_n^2)^{-1/2}e_n\})\\
&\quad = n!\, 2^n|D_i p(x)|\,(\|p\|_E^2-p^2(x))^{-1/2}(d_1^2-x_1^2)^{-1/2}\cdot\ \cdots\ \cdot(d_n^2-x_n^2)^{-1/2}\\
&\qquad \cdot(d_i^2-x_i^2)^{1/2}\cdot\operatorname{vol}(\operatorname{conv}\{0, e_1, ..., e_n\})\\
&\quad = 2^n|D_i p(x)|\,(\|p\|_E^2-p^2(x))^{-1/2}(d_1^2-x_1^2)^{-1/2}\cdot\ \cdots\ \cdot(d_n^2-x_n^2)^{-1/2}\\
&\qquad \cdot(d_i^2-x_i^2)^{1/2}\leqslant\lambda(x) \qquad \text{for}\quad j=1, ..., n \text{ and for almost every}\\
&\qquad\qquad x\in\operatorname{int}(E).
\end{aligned}$$

Here $\hat{e}_i$ denotes that the $\pm(d_i^2 - x_i^2)^{1/2} e_i$ is missing. Combining these n inequalities we obtain 4.1.

4.4. *Proof of Theorem* 4.2. Given a fat compact subset of $\mathbb{R}^n$ define

$$F_k = \left\{ x \in E\colon \operatorname{dist}(x, \partial E) \geq \frac{1}{k} \right\}$$

and

$$E_k = \bigcup_{x \in F_k} \bar{B}(x, 1/(k+1)), \text{ for } k \in \mathbb{N},$$

where $\bar{B}(x, r)$ denotes the closed euclidean ball with center at x and radius r. We have $E_k \subset E_{k+1}$ and $\operatorname{int}(E) = \bigcup \operatorname{int}(E_k)$. Moreover, the sets E_k are compact, fact, and (by 1.9) L-regular. By 4.1 and 1.12 we obtain

$$\int_{\operatorname{int}(E_k)} |\operatorname{grad} p(x)|\, dx \leq \pi^n (\deg p)\, d(0)\, \|p\|_E$$

and letting $k \to \infty$ gives 4.2.

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Note added in proof. In Theorem 4.2 it suffices to assume the set E is compact in $\mathbb{R}^n$. This follows by the fact that there exists a sequence $E_n \supset E_{n+1}$ of compact fat subsets of $\mathbb{R}^n$ such that $E = \bigcap E_n$ and each E_n has zero Lebesgue measure on ∂E.

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